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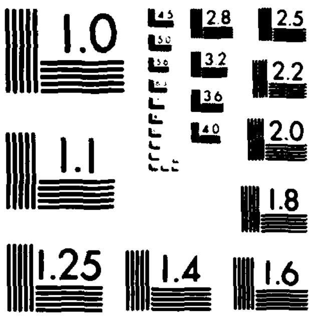


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# ON POLYNOMIAL APPROXIMATION IN THE UNIFORM NORM BY THE DISCRETE LEAST SQUARES METHOD

Lothar Reichel

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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# ON POLYNOMIAL APPROXIMATION IN THE UNIFORM NORM BY THE DISCRETE LEAST SQUARES METHOD

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ABSTRACT

The discrete least squares method is convenient for computing polynomial approximations to functions. We investigate the possibility of using this method to obtain polynomial approximants good in the uniform norm, and describe applications both to the case when the function to be approximated is known on a discrete point set only and to the case when we can freely choose the set of least squares nodes. Numerical examples are presented.

AMS (MOS) Subject Classifications: 65D10, 41A10

Key Words: Polynomial approximation, least squares method, uniform norm approximation

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## SIGNIFICANCE AND EXPLANATION

The simplicity of computing least squares polynomial approximations to functions, using function values on a discrete point set only, makes it attractive to use the least squares method for computing polynomial approximations, which are good also when the error is measured in the uniform norm. We describe how this can be done, and consider the cases

1) the function to be approximated is known on a discrete point set only. For example, we may wish to approximate a function on $[-1,1]$, but the function is only known on an equidistant point set on the interval.

2) We are free to select an arbitrary set of discrete least squares nodes. For example, we may wish to approximate an analytic function on a bounded region in the complex plane, and the function is known on the boundary of the region. We discuss how to allocate least squares nodes on the boundary without computing conformal mappings.



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# ON POLYNOMIAL APPROXIMATION IN THE UNIFORM NORM BY THE DISCRETE LEAST SQUARES METHOD

Lothar Reichel

## 1. Introduction

Polynomial approximations to functions are conveniently computed by the discrete least squares method. We describe how this method can be used to compute polynomials, which provide good approximations also when the approximation error is measured in the uniform norm on the whole domain of the function. Important is the choice of an appropriate relation between the number of discrete function values to be used, and the degree of the polynomial. This choice should depend on the distribution of the least squares nodes.

Ex. 1.1. Consider the approximation of $f(x) := (1 + 25x^2)^{-1}$ on $[-1,1]$. Use function values at $m$ equidistant points $x_k := -1 + 2\frac{k-1}{m-1}$, $k = 1(1)m$, to compute a least squares polynomial of degree $< n$.

a) $n = m$. The uniform norm approximation error increases with $m$, see Runge [5], or Dahlquist-Bjorck [4], section 4.3.4.

b) $n = \frac{\pi}{\sqrt{2}}\sqrt{m}$. The computed sequence of polynomials for increasing $n$ converges maximally to $f(x)$ in the sense of Walsh [7], i.e. the exponential rate of convergence is optimal. Details are given in section 2.

$\square$

Ex. 1.2. Regard the same approximation task as in the previous example, but use the Chebyshev nodes $x_k := \cos(\frac{2\pi k-\pi}{2m})$, $k = 1,2,\dots,m$. Already for $m = n$ we obtain a polynomial sequence which converges maximally to $f$.

$\square$

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In section 2, we state results on the rate of convergence in the uniform norm of the polynomial approximant computed by the least squares method. Our approach is to first give error bounds containing certain interpolation operator norms, and then bound the interpolation operators. Numerical examples are presented in section 3.

## 2. Estimates for the least squares operator

Let $\Gamma$ be a Jordan curve or Jordan arc in the complex plane. Denote the open set bounded by $\Gamma$ by $\Omega$. If $\Gamma$ is a Jordan arc then $\Omega$ is void. We wish to compute polynomials, which provide good approximations to functions $f$ on $\Gamma$ in the uniform norm,

$$\|f\|_u := \sup_{z\in\Gamma} |f(z)| \quad . \tag{2.1}$$

Let $\{z_{k,m}\}_{k=1}^{m}$ be a set of distinct nodes on $\Gamma$. Introduce the inner-product and corresponding semi-norm

$$(f,g) := \frac{1}{m} \sum_{k=1}^{m} f(z_{k,m})\, \overline{g(z_{k,m})} \quad , \tag{2.2}$$

$$\|f\|_m := (f,f)^{1/2} \quad , \tag{2.3}$$

where the bar denotes complex conjugation. For a given function $f$ on $\Gamma$, let $L_{n,m}f$ denote the best polynomial approximant of degree $< n$ with respect to the semi-norm (2.3), $m > n$. Let $I_n f$ be the polynomial of degree $< n$ interpolating $f$ at $n$ distinct points $\{w_k\}_{k=1}^{n}$ on $\Gamma$. We introduce the notation

$$I_n \prec L_{n,m} \quad \text{if} \quad \{w_k\}_{k=1}^{n} \subset \{z_{k,m}\}_{k=1}^{m},\ w_k \ \text{distinct} \quad . \tag{2.4}$$

We also need the definition

$$E_n(f) := \sup_{p_n} \|f - p_n\|_u \quad , \tag{2.5}$$

where the supremum is taken over all polynomials of degree $< n$.

Theorem 2.1

Define $L_{n,m}$ and $I_n$ on the set of functions continuous on $\Gamma \cup \Omega$ and analytic on $\Omega$, and let both domain and range of $L_{n,m}$ and $I_n$ be equipped with the uniform norm (2.1). (If $\Gamma$ is an arc, then $\Omega$ is empty.) Then

$$\|L_{n,m}\| < \|I_n\|(1 + \sqrt{m}) \quad \forall\, I_n \prec L_{n,m},\ \forall\, m > n \quad . \tag{2.6}$$

The growth rate of the right hand side is the smallest possible.

<u>Proof</u>. Let $P_n f$ be a polynomial of degree $< n$ such that $\|P_n f - f\|_u = E_n(f)$. Then

$$\|L_{n,m}f - f\|_m < \|P_n f - f\|_m < \|P_n f - f\|_u = E_n(f) \quad . \tag{2.7}$$

Let $I_n \dashv L_{n,m}$ and let $\ell_k(z)$ denote the Lagrange polynomials associated with $I_n$, i.e.

$$(I_n f)(z) = \sum_{k=1}^{n} f(w_k)\ell_k(z) \quad . \tag{2.8}$$

Express $L_{n,m}$ with the same polynomial basis,

$$(L_{n,m}f)(z) = \sum_{k=1}^{n} \alpha_k \ell_k(z) \tag{2.9}$$

for some constants $\alpha_k$. Substituting (2.9) into (2.7) yields

$$|\alpha_k - f(w_k)| < \sqrt{m}\, E_n(f) \quad , \quad k = 1(1)n \quad .$$

Substitution into

$$|(L_{n,m}\, f)(z)| < \sum_{k=1}^{n} |\alpha_k - f(w_k)|\ |\ell_k(z)| + \sum_{k=1}^{n} |f(w_k)|\ |\ell_k(z)|$$

yields

$$\|L_{n,m}\| = \sup_{\|f\|_u=1} \|L_{n,m}f\|_n < \|I_n\| + \|I_n\|\sqrt{m} \sup_{\|f\|_u=1} E_n(f) \quad . \tag{2.10}$$

This proves (2.6). We next give an example, where the growth rate of the bound (2.6) for increasing $m$, and $n$ held fixed, is obtained.

Let $f_q(x)$, $x \in [-1,1]$, $q > 0$, denote the piece-wise linear function

$$f_q(x) := \begin{cases} qx & , \ |x| < 1/q \\ x/|x| & , \ 1/q < |x| < 1 \quad . \end{cases} \tag{2.11}$$

Approximate $f_q$ by a 1st degree polynomial $L_{2,m}f_q$ on $[-1,1]$ for $m$ even. The polynomial $L_{2,m}f_q$ we define by least squares approximation at the $m$ points $x_1 = \frac{1}{\sqrt{2}}$, $x_2 = x_3 = \ldots = x_{\frac{m}{2}} = \frac{1}{q}$, $x_{m+1-j} = -x_j$, $j = 1(1)\,\frac{m}{2}$.

Then

$$(L_{2,m}f_q)(x) = \frac{1 + \sqrt{2}\,(\frac{m}{2} - 1)\frac{1}{q}}{1 + 2(\frac{m}{2} - 1)\frac{1}{q}2}\, x \quad .$$

Letting $q = \sqrt{\frac{m}{2} - 1}$, we obtain

$$(L_{2,m}f_q)(x) = \frac{1}{3}(1 + \sqrt{m-1})x \quad .$$

For $m$ sufficiently large,

$$(2.12) \qquad \|L_{n,m}\| > \|L_{n,m}f_q\|_u > \|L_{n,m}f_q - f_q\|_u - \|f_q\|_u = \frac{1}{3}(\sqrt{m-1} - 2) - 1 \quad .$$

Moreover,

$$(2.13) \qquad \inf_{I_2 \dashv L_{2,m}} \|I_2\| = \| \left|\frac{x-x_1}{x_m-x_1}\right| + \left|\frac{x-x_m}{x_1-x_m}\right| \|_u < \frac{2}{\pi}\ln(2) + 4 \quad \forall\, m > 4 \quad .$$

Combining (2.12) and (2.13), we see there is an $I_2$ and a constant $d > 0$, such that

$$(2.14) \qquad \|L_{2,m}\| > \|I_2\|\sqrt{m}\, d \quad \forall\, m \text{ sufficiently large} \quad .$$

The least squares nodes of this example are not distinct, but we can find distinct nodes close to those used in this example and such that $\|L_{2,m}\| > \|I_2\|\sqrt{m} \cdot d/2$.

□

Generally, we will select $n$ as an increasing function of $m$. Additional smoothness of the functions to be approximated will decrease the growth of $\|L_{n,m}\|$ with $m$, $n = n(m)$. The next theorem illustrates this.

<u>Theorem 2.2</u>

a) Let $\Gamma = [-1,1]$ and let $L_{n,m}$ have the domain $F_{d,k} :=$ $\{f : f \in C^k[-1,1], \|\frac{d^kf}{dz^k}\|_u < d\}$ equipped with norm (2.1). Then, for some constant $D$ depending on the constant $d$ and integer $k$,

$$(2.15) \qquad \|L_{n,m}\| < \|I_n\|(1 + Dm^{1/2}n^{-k}) \quad \forall\, I_n \dashv L_{n,m} \quad .$$

b) Let $\Gamma$ be a smooth Jordan curve with interior $\Omega$. Define $L_{n,m}$ on

$F_{d,k} = \{f : f$ analytic in $\Omega$, $f \in C^k(\Omega \cup \Gamma)$, $\|\frac{d^k f}{dz^k}\|_u < d\}$ equipped with

norm (2.1). Then, there is a constant $D$ such that

$$(2.16) \qquad \|L_{n,m}\| < \|I_n\|(1 + D\sqrt{m}\ |\frac{m(n)}{n}|^k) \quad , \quad \forall\ I_n \dashv L_{n,m},\ n > k+2 \quad .$$

<u>Proof</u>. The inequalities (2.15) and (2.16) follow from (2.10). To obtain (2.15) we use Jackson's theorem, see Cheney [2], p. 147. (2.16) follows from Smirnov-Lebedev [6], p. 99.

□

In the following we will assume that there is a distribution function $S(z)$ on $\Gamma$ with a strictly positive derivative for the least squares nodes $z_{k,m}$, and that these nodes are defined by

$$(2.17) \qquad S(z_{k,m}) = \frac{k-1}{m},\ k = 1(1)m,\ m = 1,2,\ldots \quad .$$

The sets of interpolation points will be subsets of the least squares node sets.

*The next theorem gives* estimates for $\|I_n\|$ for various distributions of interpolation points. We first single out two special point distributions. The definitions follow Walsh [8], ch. 7.

<u>Definitions</u>

Let $G(z)$ be the Green's function for the region exterior to $\Gamma$ with a logarithmic singularity at infinity. Let $\frac{\partial}{\partial n}$ denote the outward normal derivative. A point set $\{w_k\}_{k=1}^n$ on $\Gamma$ is called <u>uniformly distributed</u> if the $\theta_{k,n}$ defined by

$$(2.18) \qquad \int_{w_k}^{w_{k+1}} \frac{\partial G}{\partial n}(\zeta)|d\zeta| = \theta_{k,n} \quad , \quad k = 1(1)n-1 \quad ,$$

for any $0 < c < d < 1$, satisfy

$$\lim_{n\to\infty} \frac{1}{n}\ \{\text{Number of } \theta_{k,n} \text{ in } [c,d]\} = d-c \quad .$$

Integration in (2.17) is understood to be carried out along $\Gamma$ in the positive sense.

If $\theta_{k,n} = \frac{k-1}{n}$, $\forall k$, or $\theta_{k,n} = \frac{k}{n}$, $\forall k$, then the point set $\{w_k\}_{k=1}^n$ is said to be <u>equidistributed</u> on $\Gamma$.

□

Ex. 2.1. The zeros of the Chebyshev polynomial $T_n(x) = \cos(n \arccos(x))$ are equidistributed on $[-1,1]$. Equidistant nodes on $\Gamma = \{z : |z| = 1\}$ are equidistributed.

□

<u>Theorem 2.3</u>

If the interpolation points are equidistributed, and $\Gamma$ is an interval, or an analytic Jordan curve, then there is a constant $\beta$ such that

$$\|I_n\| < \frac{2}{\pi} \ln(n) + \beta, \quad n = 0,1,2,\ldots \quad . \tag{2.19}$$

If the interpolation points are uniformly distributed and $\Gamma$ is a Jordan curve or Jordan arc, then

$$\|I_n\|^{1/n} \to 1, \quad u \to \infty \quad . \tag{2.20}$$

If the interpolation points $w_k = w_{k,n}$, $k = 1(1)n$, in the limit $n \to \infty$ are distributed according to a density function $\sigma$, then

$$\|I_n\|^{1/n} = \frac{\rho_1}{\rho_0} \quad , \quad n \to \infty \quad , \tag{2.21}$$

where $\rho_1 := \exp(\sup_{z\in\Gamma} \int_\Gamma \ln|z-\zeta|\sigma(\zeta)|d\zeta|)$, and $\rho_0 :=$ $\exp(\inf_{z\in\Gamma} \int_\Gamma \ln|z-\zeta|\sigma(\zeta)|d\zeta|)$.

<u>Proof</u>.

Statement (2.19) is well-known for $\Gamma$ being an interval. For $\Gamma$ being an analytic Jordan curve, (2.19) follows from Curtis [3], theorem B. (2.20) is a special case of (2.21). The latter can be shown by potential theoretic methods. This is carried out in the appendix.

□

Approximation method

Guided by theorem 2.3, we select $n = n(m)$, so that among the least squares nodes $\{z_{k,m}\}_{k=1}^{m}$ there is a subset of $n$ points, which is uniformly distributed on $\Gamma$ as $m \to \infty$. The assumed existence of a distribution function $S(z)$, with strictly positive derivative, for the $z_{k,m}$ guarantees, that such a function $n(m)$ does exist, and, moreover, we can let $n(m) \to \infty$ as $m \to \infty$.

Maximal convergence, defined in ex. 1.1, is obtained if $\|L_{n,m}\|^{1/n} \to 1$ as $n, m \to \infty$. From (2.6), we see that a sufficient condition for maximal convergence is that the function $n(m)$ is such that $n(m) > m^{\alpha}$ for some constant $\alpha > 0$. The next theorem provides an example.

□

Theorem 2.4

Let $\Gamma = [-1,1]$, and let $z_{k,m} = -1 + \frac{2k-1}{m}$, $k = 1(1)m$. Then $n = \frac{\pi}{\sqrt{2}} \sqrt{m}$ satisfies the proposition. Conversely, there is no subset of $n = cm$ uniformly distributed nodes as $m \to \infty$, for any constant $c > 0$.

Proof. See appendix.

□

Remark

An approximation problem related to that treated in Theorem 2.4 has been discussed by Bjork [1], who measured the approximation error $L_{n,m}f - f$ with a semi-norm $\|\cdot\|_{\ell}$, with $\ell >> m$. The computation of the Euclidean norm of matrices defining the change between different orthonormal polynomial bases on $[-1,1]$, led Bjork to the suggestion that $n$ should be selected $< 2\sqrt{m}$. This choice of $n$ is close to ours, since $\frac{\pi}{\sqrt{2}} \approx 2.22$.

□

## 3. Numerical examples

When applying the approximation method, we discern the following different cases.

a) the function $f$ is known on all of $\Gamma$.

b) the function $f$ is known only on a finite point set on $\Gamma$.

c) the normal derivative of the exterior Green's function for $\Gamma$ is explicitly known

d) the normal derivative of the exterior Green's function for $\Gamma$ is not simply available.

Only if a) and c) are true, or, if we have the situation described in theorem 2.4, then we know how to select $n = n(m)$. In general we compute $L_{n,m}f$ for several $n < m$. This is illustrated in example 3.3. If then a) holds true, we can compute $\|L_{n,m}f - f\|_u$ and select the best of the computed $L_{n,m}f$'s. If instead b) is true, the selection of an appropriate $L_{n,m}f$ must be based on a numerical perturbation analysis.

Our first example is a continuation of examples 1.1 and 1.2. All computations were carried out on a AX/780 in double precision, i.e. with 15 significant digits.

Ex. 3.1. Let $\Gamma = [-1,1]$, $f(z) = (1 + 25z^2)^{-1}$, and $z_{k,m} = -1 + \frac{2k-1}{m}$, $k + 1(1)m$. Let $n$ be the largest integer $< \frac{\pi}{2}\sqrt{m}$. $I_n^T f$ below denotes the interpolation polynom $I_n f$ determined by interpolating $f$ at the zeros of $T_n(x) = \cos(n \arccos(x))$. $I_n^T f$ is known to converge maximally to $f$. In all examples on approximation on $[-1,1]$, we have used the polynomials $T_k(x)$ as basis functions.

| $m$ | $n$ | $\|L_{n,m}f - f\|_u$ | $\dfrac{\|L_{n,m}f - f\|_u^{1/n}}{\|I_n^T f - f\|_u^{1/n}}$ |
|---|---|---|---|
| 10 | 7 | 0.688 | 1.15 |
| 40 | 14 | 0.173 | 1.02 |
| 160 | 28 | $0.161 \cdot 10^{-1}$ | 1.03 |
| 640 | 56 | $0.904 \cdot 10^{-4}$ | 1.02 |
| 2560 | 112 | $0.192 \cdot 10^{-8}$ | 1.01 |

The entries of the last column are close to 1 and decrease as $n$, $m$ increase. This indicates the maximal convergence of $L_{n,m}f$ to $f$.

□

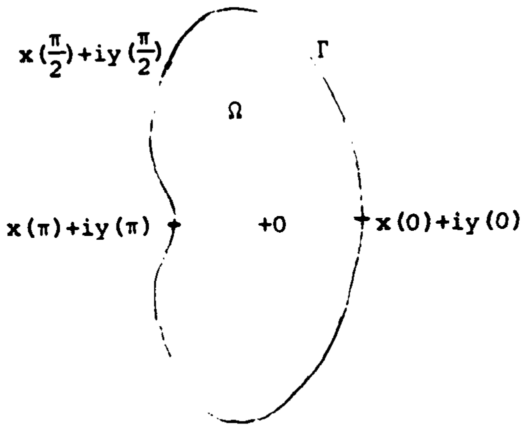
Ex. 3.2. We continue the previous example. Increase the value of $n$ to $n = 0.5m$, everything else kept as in ex. 3.1.

| $m$ | $n$ | $\|L_{n,m}f - f\|_u$ |
|---|---|---|
| 10 | 5 | 0.319 |
| 40 | 20 | 0.523 |
| 160 | 80 | 12.221 |

The error increases exponentially.

□

Ex. 3.3. Let $\Omega$ be the interior of the curve $\Gamma := \{x+iy,\ x(t) := \frac{4}{7}\cos(t) + \frac{2}{7}\cos(2t) - \frac{2}{7},\ y(t) := \sin(t) + \frac{2}{5}\sin(2t) - \frac{2}{35}\sin(4t),\ 0 < t < 2\pi\}$.




We wish to approximate functions analytic on $\Omega$ and continuous on $\Omega \cup \Gamma$ by polynomials. $\Gamma$ is somewhat similar to the ellipse $\mathcal{E} :=$ $\{z,\ z(t) = x(0)\cos(t) + iy(\frac{\pi}{2})\sin(t),\ 0 < t < 2\pi\}$.

The point set $\{z(t_{k,m})\}_{k=1}^{m}$ is equidistributed on $\mathcal{E}$ if $t_{k,m} = 2\pi \frac{k-1}{m}$, $k = 1(1)m$. This leads us to select the least squares nodes $z_{k,m} = x(2\pi \frac{k-1}{m}) + iy(2\pi \frac{k-1}{m})$, $k = 1(1)m$, on $\Gamma$. As basis functions we use the Chebyshev polynomials of the 1st kind for the interval between the foci of $\mathcal{E}$ and we scale them so that their maximum magnitude on $\Gamma$ is 1. This basis is sufficiently well-conditioned to allow representation of polynomials of a fairly high degree. Let $f(z) = (z - 2x(\pi))^{-1}$. For $m = 80$, we compute $L_{n,m}$ for $n = 30(10)70$. The best approximation so obtained is underlined.

| m | n | $\|L_{n,m}f - f\|_u$ |
|---|---|---|
| 80 | 30 | $0.193 \cdot 10^{-3}$ |
| 80 | 40 | $0.914 \cdot 10^{-5}$ |
| $\underline{80}$ | $\underline{50}$ | $\underline{0.437 \cdot 10^{-6}}$ |
| 80 | 60 | $0.250 \cdot 10^{-5}$ |
| 80 | 70 | $0.241 \cdot 10^{-2}$ . |

How overdetermined the linear system should be to yield the smallest approximation error depends on the location of the singularities of the function to be approximated. Reflecting the singularity of $f$ in the imaginary axis, we obtain $g(z) = (z - 2x(0))^{-1}$.

| $m$ | $n$ | $\|L_{n,m}g - g\|_u$ |
|---|---|---|
| 40 | 20 | $0.413 \cdot 10^{-3}$ |
| 40 | 30 | $0.114 \cdot 10^{-4}$ |
| 40 | 39 | $0.611 \cdot 10^{-6}$ |
| 40 | 40 | $0.592 \cdot 10^{-6}$ . |

$\square$


Acknowledgement

I wish to thank Germund Dahlquist for making me aware of the paper of Bjork, which aroused my interest in the approximation questions discussed. Also, I want to thank Carl de Boor for several valuable discussions during the preparation of this paper.

Appendix

Proof of theorem 2.3.

By potential theoretic methods, we prove (2.21) of theorem 2.3. Let the interpolation nodes be $\{w_k\}_{k=1}^n$, and consider the Lebesgue function

$$\Lambda_n(z) := \sum_{k=1}^{n} \prod_{\substack{j=1\\ j\neq k}}^{n} \left|\frac{z-w_j}{w_k-w_j}\right| . \tag{A.1}$$

For $z \neq w_j$, we have

$$\frac{1}{n} \ln \prod_{\substack{j=1\\ j\neq k}}^{n} |z-w_j| = \frac{1}{n} \sum_{\substack{j=1\\ j\neq k}}^{n} \ln|z-w_j| = \int_\Gamma \ln|z-w|\sigma(w)|dw| + O(\frac{\ln(n)}{n}) , \tag{A.2}$$

and

$$\frac{1}{n} \ln \prod_{\substack{j=1\\ j\neq k}}^{n} |w_k-w_j| = \int_\Gamma \ln|w_k-w|\sigma(w)|dw| + O(\frac{\ln(n)}{n}) . \tag{A.3}$$

Substituting (A.2), (A.3) into (A.1) yields

$$\Lambda_n(z) = n^2 \exp(n \int_\Gamma \ln|z-w|\sigma(w)|dw|) \cdot O(1) \cdot$$

$$\cdot \sum_{k=1}^{n} \exp(-n \int_\Gamma \ln|w_k-w|\sigma(w)|dw|), \; z \in \Gamma .$$

Hence,

$$\Lambda_n(z) = n^2 \rho_1^n \cdot O(1) \cdot n \, \rho_0^{-n} ,$$

and finally,

$$\|I_n\|^{1/n} = \|\Lambda\|_u^{1/n} = \frac{\rho_1}{\rho_0} , \quad n \to \infty .$$

If the interpolation nodes are uniformly distributed as $n \to \infty$, then $\sigma(w)$ is the equilibrium density function for $\Gamma$, and in this case $\rho_1 = \rho_0$. □

Proof of theorem 2.4.

The zeros of the Chebyshev polynomial $T_n(x) = \cos(n \operatorname{arc}\cos(x))$ are equidistributed, and therefore uniformly distributed, on $[-1,1]$. Let

$N_n^T(c,d)$ denote the number of zeros of $T_n(x)$ in the interval $[c,d]$, $-1 \leq c \leq d \leq 1$. Given a set of $n$ uniformly distributed nodes on $[-1,1]$, let $N_n(c,d)$ be their number in $[c,d]$, $-1 \leq c \leq d \leq 1$. Then $\lim\limits_{n\to\infty} \frac{1}{n} N_n(c,d) = \lim\limits_{n\to\infty} \frac{1}{n} N_n^T(c,d)$.

We first show that $n$ cannot be equal to $cm$, for any constant $c > 0$. Assume the contrary, i.e. there is a constant $c > 0$, such that as $n \to \infty$, there is a subset of $n = cm$ uniformly distributed nodes. For any $\varepsilon$ with $0 < \varepsilon < 1$, we then have

$$\lim_{n\to\infty} \frac{1}{m} (N_n(-1,-1+\varepsilon) + N_n(1-\varepsilon,1)) \leq \varepsilon \quad ,$$

and

$$\lim_{n\to\infty} \frac{1}{n} N_n(-1+\varepsilon,1-\varepsilon) = \frac{1}{\pi} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{1-t^2}}\, dt = \frac{2}{\pi} \arcsin(1-\varepsilon) \quad .$$

Therefore

$$\lim_{n\to\infty} (\frac{1}{n}(N_n(-1+\varepsilon,1-\varepsilon) + N_n(-1,-1+\varepsilon) + N_n(1-\varepsilon,1)) \leq \frac{\varepsilon}{c} + \frac{2}{\pi} \arcsin(1-\varepsilon) \quad .$$

It remains to be shown that to each $c > 0$, there is an $\varepsilon$, $0 < \varepsilon < 1$, such that $\frac{\varepsilon}{c} + \frac{2}{\pi} \arcsin(1-\varepsilon) < 1$. Let $h_1(\varepsilon) := \frac{\varepsilon}{c}$, and $h_2(\varepsilon) := 1 - \frac{2}{\pi} \arcsin(1-\varepsilon)$. Then $h_1(0) = h_2(0) = 0$. Further $h_1'(\varepsilon) = \frac{1}{c}$, $h_2'(\varepsilon) = \frac{2}{\pi} \frac{1}{\sqrt{2\varepsilon-\varepsilon^2}}$, $\varepsilon > 0$. In a punctured neighborhood of $\varepsilon = 0$, we have $h_2'(\varepsilon) > h_1'(\varepsilon)$. This shows that $h_2(\varepsilon) > h_1(\varepsilon)$ in that neighborhood, and we are ready.

Next we show that a subset of $\frac{\pi}{\sqrt{2}} \sqrt{m}$ nodes can be distributed uniformly with respect to $\frac{\partial G}{\partial n}(z) = \frac{1}{\pi} \frac{1}{\sqrt{1-z^2}}$, $z \in [-1,1]$. The distance between the last two largest zeros $x_{1,n}$, $x_{2,n}$ of $T_n(x)$ is

$$x_{1,n} - x_{2,n} = \cos(\frac{\pi}{2n}) - \cos(\frac{3\pi}{2n}) = 2\sin(\frac{\pi}{n})\sin(\frac{\pi}{2n}) = \frac{\pi^2}{n^2} + O(\frac{1}{n^4}) \quad .$$

With $m$ equidistant nodes, $z_{k,m} = -1 + \frac{2k-1}{m}$, $k = 1(1)m$, we have

$z_{k+1,m} - z_{k,m} = \frac{2}{m}$. We require

(A.4) $$\frac{2}{m} = \frac{\pi^2}{n^2},$$

or equivalently $n = \frac{\pi}{\sqrt{2}} m$. Since the zeros of $T_n(x)$ are most dense at the ends of the interval, the choice (A.4) of $n$ guarantees that for every $m$ a subset of $n$ nodes can be selected which is close to the set of zeros of $T_n(x)$, and uniformly distributed with respect to $\frac{\partial G}{\partial n}$ as $n \to \infty$.

□

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The discrete least squares method is convenient for computing polynomial approximations to functions. We investigate the possibility of using this method to obtain polynomial approximants good in the uniform norm, and describe applications both to the case when the function to be approximated is known on a discrete point set only and to the case when we can freely choose the set of least squares nodes. Numerical examples are presented.

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